PROPAGATION OF THERMAL WAVES FROM THE BOUNDARY OF TWO MEDIA

(RASPROSTRANIYE TEPLOVOI VOLNY OT GRANITSY DVUKH SRED)

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The problem of the propagation of heat from a plane boundary, and the decay of the discontinuity of temperature is considered, taking account of the change in phase of the material. An evaluation is carried out of the energy propagated into a medium with small thermal conductivity upon instantaneous evolution of heat at a point on the boundary of separation of two media.

1. Suppose that on the plane $x = 0$, (see Fig. 1), the temperature T_0 is maintained. After being heated up to a temperature $T_* < T_0$, the

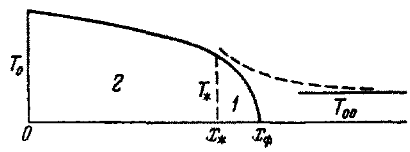

Fig. 1.

material is transformed into another phase state (region 2 in Fig. 1). The heat capacity c and the coefficient of thermal conductivity κ for $T < T_*$ are labeled by the index 1 (region 1 in Fig. 1), and for $T > T_*$ by the index 2.

The law of heat conduction is described by the equations

$$c_i \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \varkappa_i(T) \frac{\partial T}{\partial x}, \qquad \varkappa_i = \varkappa_{0i}\varphi_i(z), \qquad \left(z = \frac{T}{T_0}\right) \quad (i=1, 2) \tag{1}$$

At the boundary of phase transition x_*, the condition of heat balance is written as:

$$\varkappa_2 \frac{\partial T}{\partial x}\bigg|_{x_*-0} - \varkappa_1 \frac{\partial T}{\partial x}\bigg|_{x_*+0} = \lambda \frac{dx_*}{dt} \tag{2}$$

The problem is characterized by parameters, the units of which are expressed through the units of length L, time t, temperature T, and the quantity of heat Q:

$$c_i = QL^{-3}T^{-}, \qquad \varkappa_{0i} = QT^{-1}L^{-1}t^{-1}, \lambda = QL^{-3}, \qquad T_0 = T_* = T_1 = T$$

From these parameters, the coordinate x and the time t, it is possible to form only one independent dimensionless variable and a series of dimensionless constants:

$$\eta_i = x\left[\frac{c_i}{2\varkappa_{0i}t}\right]^{1/2}, \qquad \gamma = \left[\frac{c_1\varkappa_{10}}{c_2\varkappa_{20}}\right]^{1/2}, \qquad \mu = \frac{\lambda}{C_2T_0}, \qquad \beta = \frac{T_*}{T_0} \tag{3}$$

Therefore, equation (1) is written as:

$$\frac{d}{d\eta_i}\varphi_i(z_i)\frac{dz_i}{d\eta_i} + \eta_i\frac{dz_i}{d\eta_i} = 0 \tag{4}$$

If the initial temperature of the medium $T_{00} = 0$, the boundary conditions fixing z_1 and z_2 the front of the thermal wave x_Φ and the boundary of the phase transition x_*, will be

$$z_1(\eta_{1\Phi}) = 0, \qquad z_1(\eta_{1_*}) = \beta, \qquad z_2(0) = 1, \qquad z_2(\eta_{2*}) = \beta$$

$$\varphi_2(z_2)\frac{dz_2}{d\eta_2}\Big|_{\eta_{2*}} - \gamma\varphi_1(z_1)\frac{dz_1}{d\eta_1}\Big|_{\eta_{1_*}} = \mu\eta_{2_*} \tag{5}$$

$$\varphi_2(1)\left(\frac{dz_2}{d\eta_2}\right)_0 + \int_0^{\eta_{2_*}} z_2\,d\eta_2 + \gamma\int_{\eta_{1_*}}^{\eta_\Phi} z_1\,d\eta_1 + \mu\eta_{2_*} = 0$$

Integral curves of equation (4) were investigated in detail by Barenblatt in reference [3], from the data of which it follows that there exists a unique integral curve satisfying at the end of the distance $x_\Phi(t)$ the condition $T_\Phi = \kappa\,(\partial T/\partial x)_\Phi = 0$. If $T_{00} \neq 0$, the solution is given by integral curves having at infinity a horizontal asymptote [5].

In analogous fashion, the problem of the decay of the temperature discontinuity is considered. If it be assumed that at the moment $t = 0$, two half-planes with temperatures T_1 and T_{II} are put in contact, the solution is self-similar. Due to self-similarity, the temperature on the boundary $x = 0$ remains constant. It should be determined from the condition of continuity of the flow of heat at the section $x = 0$, for example, after solving the preceding problem for $x > 0$ and $x < 0$ for its dependence on the parameter T_0.

2. We consider the problem of the instantaneous generation of heat on the boundary between two media with zero initial temperatures. We assume that $\kappa_i = \kappa_{0i}T^{k-1}$. If $\kappa_{01} = \kappa_{02}$, $c_1 = c_2$, then in the plane case [1] the solution can be written:

$$\xi_1 = x\left[Q^{1-k}\frac{ck}{\varkappa_0 t}\right]^{\frac{1}{k+1}}, \qquad T = \left(\frac{ckQ^2}{\varkappa_0 t}\right)^{\frac{1}{k+1}}\left[\frac{k-1}{2k(k+1)}(e_0 - \xi_1^2)\right]^{\frac{1}{k-1}}$$

$$e_0 = \left\{2\left[\frac{2k(k+1)}{k-1}\right]^{\frac{1}{k-1}}\Gamma\left(\frac{1}{2}+\frac{k}{k-1}\right)\left[\Gamma\left(\frac{k}{k-1}\right)\Gamma\left(\frac{1}{2}\right)\right]^{-1}\right\}^{\frac{2(k-1)}{k+1}} \tag{6}$$

The condition of equality of temperature and heat flow in the plane $x = 0$ for $\kappa_{01} \neq \kappa_{02}$, $c_1 \neq c_2$ requires that the energy should be distributed according to the law

$$Q_1 / Q_2 = \gamma, \qquad Q_1 + Q_2 = Q \tag{7}$$

Thereupon the solution is given by formulas (6), in which it is necessary to set $Q = 2Q_2$ for $x > 0$, and $Q = 2Q_1$ for $x < 0$. It is interesting to compare formula (7) with the analogous one in the case of gas dynamics [1], when the redistribution of energy on the discontinuity at the boundary of two media is governed by the law $E_2/E_1 = (\rho_1/\rho_2)^{1/2}$. We notice, however, that for $k_1 > k_2$, it is possible to generate energy instantaneously only in medium *1*. The problem of the instantaneous generation of heat from a point on the boundary of two media ($k_1 = k_2$) depends only on

$$\xi = rt^{\frac{1}{1-3k}}\left[\frac{Q}{c\psi(k)}\right]^{\frac{1-k}{3k-1}}\left(\frac{ck}{\varkappa_0}\right)^{\frac{1}{3k-1}} = rt^{\frac{1}{3k-1}}B^{-1}(k)$$

$$\psi(k) = 2\pi\left[\frac{k-1}{2k(3k-1)}\right]^{\frac{1}{k-1}}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{k}{k-1}\right)\left[\Gamma\left(\frac{3}{2}+\frac{k}{k-1}\right)\right]^{-1}$$

and the angle θ. The quantity of heat introduced into medium *2* is conserved in time:

$$\Delta Q\Psi(k) = 2\pi Q\int_{\pi/2}^{\pi}\sin\theta\, d\theta\int_0^{\xi_\Phi(\theta)} F(\xi,\theta)\,\xi^2\, d\xi, \quad T = \left[\frac{ck}{\varkappa_0}\left(\frac{Q}{c\Psi(k)}\right)^{1/3} t^{-1}\right]^{3/(3k-1)} F(\xi,\theta) \tag{8}$$

In this, there exists a surface through which no heat flows.

If $k_1 \neq k_2$, the problem is not self-similar. However, in this case also it is possible to evaluate approximately the quantity of heat transferred into medium *2*. We shall assume that in medium *1* the region of high temperature and the radius of the heated hemisphere change according to the law

$$r_\Phi = B(k_1)\, t^{\frac{1}{3k_1-1}}, \qquad T_0 = \frac{3Q}{4\pi c r_\Phi^3} = At^{\frac{3}{1-3k_1}} \tag{9}$$

Assuming that $\kappa_2 \ll \kappa_1$, when it may be considered that in medium *2* the heat is transmitted by means of plane waves, we obtain

$$\Delta Q = 2\pi \int_0^{r_\Phi} r dr \int_t^\tau \varkappa_2 [T_0(t)] \left(\frac{\partial T}{\partial x}\right)_{x=0} dt, \quad \tau = \left(\frac{r}{B}\right)^{3k_1-1} \qquad (10)$$

The value of the derivative $(\partial T / \partial x)_{x=0}$ is determined from the solution of the problem of the propagation of heat from a wall [2,5], the temperature of which is

$$T = A(t_1 + \tau)^{3/(1-3k_1)}, \qquad 0 \leqslant t_1 \leqslant t - \tau \qquad \text{for } \Delta Q \ll Q.$$

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